

Poincare-Cartan form for scalar fields in curved background.

Pankaj Sharan

Physics Department, Jamia Millia Islamia,
New Delhi 110 025, INDIA

Abstract

Poincare-Cartan form for scalar field is constructed as a differential 4-form in a ‘directly Hamiltonian’ formalism which does not use a Lagrangian. The canonical momentum p of a scalar field ϕ is a 1-form and the Poincare-Cartan 4-form Θ is $(*p) \wedge d\phi - H$ where the Hamiltonian H is a suitable 4-form made from ϕ and p using the Hodge star operator defined by the Riemannian metric of the background spacetime. An allowed field configuration is a 4-dimensional surface in the 9-dimensional extended phase space such that its tangent vectors annihilate $\Omega = -d\Theta$. Relation of this to variational principle, symmetry fields and conserved quantities is worked out. Observables are defined as differential 4-forms constructed from field and momenta smeared with appropriate test functions. A bracket defined by Peierls long ago is found to be the suitable candidate for quantization.

1 Introduction

The Hamiltonian formulation is basic to quantum theory. Despite this, the quantum theory of *fields* always begins with a Lagrangian. The main reason for this is, in words of P. A. M. Dirac [1] “it is not at all easy to formulate the conditions for a theory to be relativistic in terms of the Hamiltonian”. The conditions for relativistic invariance are satisfied by choosing a Lagrangian function to be a relativistic scalar. This function can be constructed as a scalar by balancing indices of vector, tensor or spinor fields and their four-dimensional derivatives.

The purpose of this series of papers is to show how one can directly set up a Hamiltonian formalism for relativistic fields, including fields in arbitrary curved background, without first writing a Lagrangian and then proceeding to the Hamiltonian through the Legendre transformation. A Hamiltonian formalism can be set up in terms of fields and their canonical momenta quite as easily as a Lagrangian is written in terms of fields and their derivatives provided we treat fields and canonical momenta as differential forms (with values in spaces that characterize them). The canonical momenta in our formalism are differential

forms of *one degree higher than the fields*. Thus, mathematically, coordinates and their momenta are not quantities of the same type. This fundamental change in the mindset allows us to set up a covariant coordinate-free formalism which is Hamiltonian from the very beginning and does not require a Lagrangian for its definition.

Preliminary work in this direction already exists in formalisms variously known as ‘finite-dimensional covariant formalism’ or ‘multisymplectic’ or ‘polysymplectic’ formalism. The basic idea was given by Weyl and de Donder in the so-called ‘multiple integral problem in the calculus of variations’ and was developed by Kastrup, Kanatchikov, Gotay et al and Rovelli and others. See references and discussion at the end of this section.

It is commonly believed that the Hamiltonian formalism singles out time as a special variable and this spoils the relativistic invariance which would have required space and time to be treated on the same footing. This is true if one regards derivative of fields with respect to the time coordinate as fundamentally different from that with respect to a space coordinate. But if we treat all the four derivatives $\partial_\mu \phi$ of a scalar field ϕ as one quantity then it follows we should allow four components p_μ of momenta to be associated with this one field variable. We should not pair one coordinate with one momentum degree of freedom. Such a pairing is a peculiarity of the Hamiltonian mechanics based on a single evolution parameter time, whereas fields extend in space as well as time.

The fundamental principle in classical mechanics is that variation of a quantity called action is zero. The laws of nature allow only those configurations of physical variables which achieve an extremum for action. And this requirement of extremum is the classical limit when \hbar is regarded as small.

For classical mechanics time can be regarded as a ‘base manifold’ and coordinates and momenta are in the ‘fibre’. This is the extended phase space. Action is an integrated value $\int \Theta$ of a one-form called Poincare-Cartan form $\Theta = pdq - Hdt$ on a supposed trajectory in extended phase space. The variation of the trajectory is determined by a vector field of infinitesimal displacements. The condition that the Lie derivative of the action along the proposed trajectory with respect to the field of variation is zero when the the coordinates are fixed at the ends of the trajectory determines the trajectory. The tangent vectors to allowed trajectories determine Hamiltonian vector fields. The variational principle can be reformulated by saying that the Hamiltonian vector fields of allowed trajectories annihilate the differential 2-form $\Omega = -d\Theta$ where Θ is the Poincare-Cartan (PC) form.[2]

In order to set up a purely Hamiltonian formalism for fields, we must first try to define a suitable PC-form for fields.

The PC-form Θ has two parts : the so-called fundamental form pdq which governs geometry of the phase space, and the Hamiltonian part $-Hdt$ which determines the dynamics for the given system.

The logic for writing a PC-like form for a scalar field goes like this. In field theory, the field ϕ is the configuration variable analogous to q . Time and space are four “time” variables t^μ , $\mu = 0, 1, 2, 3$. We purposely use the letter t also for space coordinates t^i , $i = 1, 2, 3$ to emphasize this point. We expect the PC-form

for fields to be a differential *four*-form whose spacetime integral will give the quantity we call action. For a single scalar field, the momenta are related to velocities by $p_\mu = \partial_\mu \phi$. Thus we can keep them together as a 1-form $p = p_\mu dx^\mu$. To get a fundamental 4-form similar in appearance to pdq we need a 3-form (and not a 1-form p) to be multiplied to $d\phi$.

There is a natural way to produce a 3-form out of a 1-form, namely, by using the metric of the spacetime through the star-dual $*p$. We are led naturally to introduce the following expressions for the PC 4-forms :

$$\Theta = (*p) \wedge d\phi - H \quad (1)$$

where H is a differential 4-form

$$\begin{aligned} H &= \frac{1}{2}(*p) \wedge p + \frac{1}{2}m^2\phi^2(*1) \\ &= \left(-\frac{1}{2}\langle p, p \rangle + \frac{1}{2}m^2\phi^2 \right) (*1). \end{aligned} \quad (2)$$

We have used the definition of the star operator relating it to the inner product determined by $g_{\mu\nu}$ which has a signature corresponding to $(-, +, +, +)$. Our convention for the star operator is the same as Sharan[3] or Choquet-Bruhat and DeWitt-Morette[4], and is very briefly summarized in Appendix A.

Observe that the Hamiltonian 4-form H is defined solely in terms of the field variable ϕ and the momenta p (or $*p$). It is a coordinate independent definition. H is a 4-form and it should not be confused with the Hamiltonian density or the energy density of the usual Lagrangian field theory. (That density is a 3-form which will be seen to be the conserved quantity for time translations in static spacetimes.)

A field theory involves infinitely many degrees of freedom. The traditional view is to think of each value $\phi(\mathbf{x}, t)$ for space points \mathbf{x} on a plane of constant time t as a separate degree of freedom for a scalar field. This is the usual ‘3+1’ Hamiltonian point of view. See Chernoff and Marsden[5] for a rigorous account of Hamiltonian systems of infinitely many degrees of freedom.

There is another, more interesting way to look at this. One can regard a solution of the field equations as a section or a surface in the *finite* dimensional extended phase space four of whose coordinates are the spacetime coordinates. The infinitely many ways in which this surface can be embedded in the extended phase space is a reflection of the infinitely many degrees of freedom of the field system.

For our example, the extended phase space for a single scalar field is a nine-dimensional manifold (four spacetime variables t^μ , one field variable ϕ and four momentum variables in p). A possible configuration of the field (that is, a solution of the field equations) is a four dimensional surface in this nine dimensional space “above” the four dimensional spacetime. The fiber bundle picture is helpful because we are interested in ‘sections’ or functions from spacetime base into the fields and momenta. Mathematically, there may be more general submanifolds or surfaces in the extended phase space but they do not seem to be physically relevant.

As mentioned above the mathematical formalism of the present paper is similar to the “multisymplectic” Lagrangian approach to field theory in the works of Le Page, as reviewed and developed by Kastrup[6], the De Donder-Weyl[7] approach of Kanatchikov[8] and the covariant Hamiltonian-Jacobi formalism of Rovelli[9]. Recent contributions to multisymplectic formalism are by Gotay and collaborators[10]. Our approach is different from these because we use the background spacetime metric in an essential way through the Hodge star operator. Also, we treat the spacetime degrees of freedom t^μ which specify the base differently from the field or momentum degrees of freedom which are in the fibre above the base. We require the PC-form to be a 4-form whose first term is linear in $d\phi$ to imitate pdq term and the second term is a 4-form $-H$ proportional to volume form $(*1)$. If, for instance, there are two fields ϕ_1 and ϕ_2 , a 4-form involving a factor $d\phi_1 \wedge d\phi_2$ is possible in principle but that does not seem to be allowed in the formalism for matter fields. Similarly other ‘non-canonical’ expressions are possible in place of the standard $pdq - Hdt$ like expression. For gravity, the Einstein-Hilbert PC form does seem to have a non-standard expression as we shall see in a later paper. But gravity is a special case anyway. For gravity the ‘internal’ degrees of freedom in the fibre related to arbitrary choice of local inertial frames and spacetime bases which define the transformation of all field and momenta differential forms happen to coincide.

It is natural and tempting to put our formalism in the fibre bundle language, but we avoid that for the sake of clarifying the physical concepts. For most part we assume the bundle to be a direct product of spacetime and the fibre manifold.

Our aim is to develop a purely Hamiltonian approach and define a suitable bracket to help build a quantum theory. The only reliable way to convert a classical theory into a quantum theory is to define a suitable antisymmetric (or symmetric) bracket for observables of the theory which can be re-interpreted in quantum theory as a commutator (or anticommutator). Our phase space has a very different character than the traditional phase space and our coordinate and momenta are differential forms of different degrees. In the traditional formalism the observables are real valued functions on the phase space and the definition of the Poisson bracket uses the pairing of one coordinate with one canonical momentum degree of freedom. But that is special to one-time formalism of mechanics.

But in mechanics there is another way to look at the Poisson bracket. The bracket $\{B, A\}$ of two observables A and B refers to the rate of change of one observable B when the other observable A acts as the Hamiltonian. In one-time formalism the rate of change of a quantity is mathematically the same type of quantity as the original quantity. When space *and* time are evolution parameters then the rate of change can only mean rate of change along a vector field. This rate of change is the Lie derivative. Thus we need the Lie derivative of one quantity with respect to the Hamiltonian vector field determined on the phase space by the other quantity.

Whereas the Hamiltonian vector field for any observable exists for in mechanics the same may not be so for fields. We find that the concept of a covariant bracket introduced by Peierls[11] in 1952 (and promoted extensively by De

Witt[12]) is a natural object to use in our Hamiltonian theory of fields. Here the rate of change of one quantity is taken when *the other quantity is added to the Hamiltonian as an infinitesimal perturbation* and vice-versa. The Poisson brackets of mechanics can be defined without reference to any Hamiltonian whereas the Peierls bracket requires the existence of a suitable governing Hamiltonian. Roughly speaking, the Poisson bracket can be described as the “equal time” Peierls bracket with zero Hamiltonian.

This gives us added insight into the Hamiltonian mechanics of one time formalism, particularly the concept of causality in systems with time dependent Hamiltonians. The interesting features for one-time formalism of classical mechanics relating to causality and Peierls bracket which are revealed by our formalism of fields will be published elsewhere.

In section II we define the Poincare-Cartan form. We set up the variational principle and Noether’s theorem in sections III and IV. We define our observables as smeared 4-forms and their Peierls bracket in section V. Symmetries and conserved quantities are discussed in section VI and VII and the Hamilton-Jacobi formalism is discussed briefly in section VIII. Notation is summarized in appendix A. A calculation for the solution manifold in section II is outlined in appendix B.

2 Poincare-Cartan form for a scalar field

For fields the extended phase space is a bundle with the four-dimensional spacetime T as base space. We denote the spacetime points by $t = (t^0, t^1, t^2, t^3) \in T$. Let us consider the one-dimensional fibre of 0-forms with coordinate ϕ and the four-dimensional fibre of 1-forms whose points are labelled by $p = p_\mu dt^\mu$. We can think of the extended phase space Γ to be the base (of spacetime) with a five-dimensional fibre at each point which is a direct sum of 0-forms and 1-forms.

We require the momentum canonical to a scalar field ϕ to be a 1-form $p = p_\mu dt^\mu$ where coefficients p_μ are independent variables. The PC-form on this nine-dimensional extended phase space (with coordinates t^μ, ϕ, p_μ) is chosen as

$$\Theta = (*p) \wedge d\phi - H \quad (3)$$

where H is a 4-form constructed from p and ϕ . The simplest choice is a Hamiltonian with a ‘kinetic energy term’ and a ‘mass term’ :

$$\begin{aligned} H &= \frac{1}{2}(*p) \wedge p + \frac{1}{2}m^2\phi^2(*1) \\ &= \left(-\frac{1}{2}\langle p, p \rangle + \frac{1}{2}m^2\phi^2 \right) (*1). \end{aligned} \quad (4)$$

It is necessary to point out here that although our star operator is limited to the four-dimensional spacetime the exterior derivative works in the nine-dimensional extended phase space. Thus $d\phi$ is linearly independent of dt^μ and so also independent of $p = p_\mu dt^\mu$. The coefficients p_μ are independent coordinates. Therefore dp_μ are linearly independent of $d\phi$ and dt^μ .

It is also worth pointing out that the definition of star operator requires the existence of a set of orthonormal basis fields with a given orientation. This is where gravity sneaks in as a universal field. In the present paper the gravitational field will be fixed as an external field defining the star operator.

Dynamics is determined by the 5-form

$$\Omega = -d\Theta = -(d * p) \wedge d\phi + dH, \quad (5)$$

and the variational principle can be stated as follows :

The solution manifold σ in the extended phase space is a section whose tangent vectors annihilate Ω .

This statement is explained below. The relation of this statement of variational principle to the usual statement for variation of the action is discussed in the next section.

In mechanics we look for phase trajectories. Here, in field theory we look for a four-dimensional image of a *section*, that is, a mapping σ from the four dimensional base into a 4-dimensional submanifold of the nine-dimensional extended phase space :

$$\sigma : t = \{t^\mu\} \rightarrow \{t^\mu, \phi = F(t), p = G_\mu(t)dt^\mu\}. \quad (6)$$

By abuse of language we will denote the mapping as well as its image of the base by the same letter σ . The context will make it clear what the symbol corresponds to.

σ defines a surface or sub-manifold such that if X_0, X_1, X_2, X_3 are four linearly independent vectors in the tangent space of this submanifold at any point then the 1-form obtained by the interior product of all these with Ω should be zero :

$$i(X_3)i(X_2)i(X_1)i(X_0)\Omega = 0. \quad (7)$$

Recall that the interior product of a vector field X with an r -form α is defined as the $(r-1)$ -form $i(X)\alpha$ so that $i(X)\alpha(Y_1, \dots, Y_{r-1}) = \alpha(X, Y_1, \dots, Y_{r-1})$. Depending on typographical convenience we shall denote the interior product of a vector field X with a form α by $i(X)\alpha$ or $i_X\alpha$.

The meaning of variational principle above is that for arbitrary vector field Y on Γ ,

$$\Omega(X_3, X_2, X_1, X_0, Y) = 0. \quad (8)$$

In the following we call σ determined by this condition as a “solution submanifold”. We can choose X_μ to be just the push-forwards by σ of the coordinate basis vectors $\partial_\mu \equiv \partial/\partial t^\mu$:

$$X_\mu = \sigma_*(\partial_\mu) = \partial_\mu + F_{,\mu}\partial_\phi + G_{\nu,\mu}\partial_{p_\nu}.$$

For our case Ω can be calculated easily. Using

$$d(*p \wedge p) = d(*p) \wedge p + *p \wedge (dp) = 2d(*p) \wedge p$$

we get

$$dH = (d * p) \wedge p + m^2 \phi d\phi \wedge (*1)$$

Substituting in Ω we see that it *factorizes*

$$\Omega = (d * p - m^2 \phi (*1)) \wedge (p - d\phi), \quad (9)$$

where we use the fact that the 5-form $(*1) \wedge p$ in four variables t is zero because there are five factors of dt 's.

We give details of the calculation for flat space in Appendix B. The condition on F and G_μ to define a solution manifold is

$$G_\mu = F_{,\mu}, \quad d * dF - m^2 F(*1) = 0 \quad (10)$$

which is the solution $\phi = F(t)$ to the Klein-Gordon equation for the field ϕ .

There is a less rigorous but physically straightforward way to see what solution manifold should be. Vector fields annihilating $p - d\phi$ imply $p_\mu = \partial_\mu \phi$. Similarly, for $p = d\phi$, the first factor gives zero if

$$d * d\phi - m^2 \phi(*1) = 0 \quad (11)$$

Now

$$\begin{aligned} d * d\phi &= \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi) dt^0 \wedge \dots \wedge dt^3 \\ &= \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi) (*1) \end{aligned} \quad (12)$$

and thus ϕ satisfies the Klein-Gordon equation with the Laplace-Beltrami operator of the curved space.

We close this section with a few remarks.

1. Since ϕ and t^μ are coordinates in the extended phase space, $d\phi$ and dt^μ are linearly independent. Therefore $p_\mu dt^\mu - d\phi = 0$ is meaningless as it stands. What it implies is that there exists a subspace or submanifold σ of the nine-dimensional extended phase space such that any of the independent vector fields X_μ tangent to σ satisfies

$$(p_\mu dt^\mu - d\phi)(X) = 0.$$

The 1-form $p_\mu dt^\mu$ has non-zero coefficients for the dt^μ 's and zero for $d\phi$ and dp_μ . These coefficients p_μ themselves are independent coordinates. Thus, although $*p \wedge p = -\langle p, p \rangle (*1)$ is proportional to 4-form $*1$ its exterior derivative $d(*p \wedge p)$ need not be zero.

2. If there are several fields ϕ^a then we can construct the PC-form similarly as

$$\Theta = *p_a \wedge d\phi^a - H \quad (13)$$

where $p_a = p_{a\mu} dt^\mu$ are canonical momenta for the fields ϕ^a and H is a 4-form depending on all the fields and the momenta.

3. If (ϕ_1, p_1) and (ϕ_2, p_2) are two solutions for the scalar field, then the 4-form

$$\begin{aligned}
& d(\phi_1 * p_2 - \phi_2 * p_1) \\
&= d\phi_1 \wedge *p_2 + \phi_1 d * p_2 - (1 \leftrightarrow 2) \\
&= (d\phi_1) \wedge (*d\phi_2) + \phi_1 (m^2 \phi_2) * (1) - (1 \leftrightarrow 2) \\
&= (d\phi_1) \wedge (*d\phi_2) - (d\phi_2) \wedge (*d\phi_1) \\
&= 0
\end{aligned}$$

because, (using the identity $(*) \wedge s = (-1)^{r(n-r)} t \wedge (*s)$ for any r -forms t and s in an n -dimensional space) we conclude that in our case

$$(d\phi_1) \wedge (*d\phi_2) = -(*d\phi_1) \wedge (d\phi_2) = (d\phi_2) \wedge (*\phi_1).$$

Thus, by Stokes theorem the integral

$$\oint (\phi_1 * p_2 - \phi_2 * p_1)$$

over any closed surface is zero. This leads to a linear space of solutions on which there is a time-independent scalar product.

3 Stationary Action

We have seen that a specific solution to the field equations can be realized as a four-dimensional submanifold σ of the nine-dimensional extended phase space.

Hamilton's variational principle involves comparing the integral of the PC-form on a proposed four-dimensional solution submanifold with a similar integral on a neighboring submanifold.

Let $\sigma : t \rightarrow \{\phi = F(t), p_\mu = F_{,\mu}\} \in \Gamma$ be the submanifold corresponding to some given solution.

Let D be a region of spacetime and ∂D its boundary. Calculate the PC-form Θ on the region $\sigma(D)$ of the extended phase-space mapped by σ .

Let Y be a vector field of variation. We can paraphrase Arnold's elegant argument[13] for mechanics and apply to fields. Calculate the Lie derivative using the formula $L_Y = i_Y \circ d + d \circ i_Y$ (see for example [14]) :

$$\begin{aligned}
\delta_Y \int_{\sigma(D)} \Theta &\equiv L_Y \int_{\sigma(D)} \Theta \\
&= \int_{\sigma(D)} L_Y \Theta \\
&= \int_{\sigma(D)} (i_Y \circ d + d \circ i_Y) \Theta
\end{aligned}$$

$$\begin{aligned}
&= - \int_{\sigma(D)} i_Y \Omega + \int_{\sigma(D)} d[i_Y \Theta] \\
&= \oint_{\partial\sigma(D)} i_Y \Theta
\end{aligned}$$

where the integral of $i_Y \Omega$ on the submanifold σ is zero because the integral evaluates $i_Y \Omega$ on tangent vectors $\sigma_*(\partial_\mu)$ to the proposed solution submanifold which is zero. Thus variational principle can also be expressed as,

$$\delta_Y \int_{\sigma(D)} \Theta = \oint_{\partial\sigma(D)} i_Y \Theta \Big|_0. \quad (14)$$

Here we use the symbol 0 to denote a quantity “on-shell”, that is, evaluated on a solution submanifold. It needs to be emphasized that since the variation field Y is not restricted to the solution surface, it will be a mistake to use $\phi = F, p_\mu = F_{,\mu}$ before the evaluation of $i_Y \Theta$.

Since Θ involves $d\phi$ and dt^μ (and no dp_μ), the surface integral of 3-form $i(Y)\Theta$ gives zero if the infinitesimal field Y is zero along the directions $\partial/\partial t^\mu$ and $\partial/\partial\phi$. But there is no restriction on variation in momenta directions.

We can re-express the variational principle (or principle of stationary action) in extended phase space as :

Under variation by a field Y with ϕ, t^μ held fixed at the boundary the action evaluated at the solution submanifold σ is stationary :

$$\delta_Y \int_{\sigma(D)} \Theta = 0. \quad (15)$$

4 Noether's Theorem

Let us consider a variation Y not necessarily zero at the boundary $\sigma(D)$ where σ is solution manifold. Equation (14) for variations is

$$\delta_Y \int_{\sigma(D)} \Theta = \int_{\sigma(D)} L_Y \Theta = \oint_{\partial\sigma(D)} i_Y \Theta \Big|_0 \quad (16)$$

If we know that for some given type of variation Y ,

$$L_Y \Theta = 0 \quad (17)$$

then we say that action is invariant under the infinitesimal mapping represented by the fields Y and Y is called a ‘symmetry field’. Usually, the symmetry fields satisfy the conditions $L_Y(*p \wedge d\phi) = 0$ and $L_Y H = 0$ separately. The surface integral

$$\oint_{\partial\sigma(D)} i_Y \Theta \Big|_0 = 0 \quad (18)$$

gives a conservation law for the 3-form $i_Y\Theta$. In the particular case when the boundary ∂D is constituted by two spacelike surfaces, the 3-form $i_Y\Theta$, restricted to either surface represents the volume density of the conserved “charge” on that surface.

5 Observables and Peierls bracket

Our formalism treats coordinate ϕ and its canonical momentum p respectively as 0- and 1-forms. In classical mechanics they seem to be quantities of the same type because in one-dimensional base manifold representing time, 0-forms and 1-forms are both 1-dimensional spaces. This situation changes for field theory in four dimensions. There 0- and 1-forms are respectively spaces of one and four dimensions.

The observables of our theory are quantities like action : integrated quantities over a four dimensional submanifold. A typical observable is an integrated 4-form $A = \int \alpha$. The support of α , that is set over which it has non-zero values could be suitably restricted to allow for local quantities as observables. For example, the scalar field ϕ is related to the observable $\int \phi j(*1)$ where $j(t)$ is a scalar ‘switching function’ which is non-zero in a small spacetime region. For simplicity we would call both the integrated as well as the non-integrated quantity by the same name ‘observable’, and it leads to no confusion.

The Peierls bracket is the natural bracket-like quantity in this formalism. When the Hamiltonian 4-form H is perturbed by observable λB (where λ is an infinitesimal parameter) the solution manifold shifts, and, after taking causality into account, the difference between the two solutions at different points in the limit of $\lambda \rightarrow 0$ determines a ‘vertical’ vector field X_B . This field changes all other observables. The change in an observable A is equal to the Lie derivative $D_B A \equiv L_{X_B} A$ of A with respect to X_B . Switching the roles of B and A we can calculate $D_A B$. The Peierls bracket $[A, B]$ is defined as the difference $D_B A - D_A B$.

For illustration we outline the calculate the Peierls bracket for the scalar field with itself in Minkowski space. The observable in question is the integrated 4-form

$$B = \int \beta = \int \phi j(*1)$$

where j is a switching function in spacetime with which the field ϕ is ‘smeared’. The Hamiltonian is changed to $H + \lambda B$ and the solution manifold given by $t \rightarrow \phi = F_0(t), p_\nu = F_{0,\nu}$ gets modified to a solution manifold which is determined by the 5-form

$$\begin{aligned} \Omega_B &= -d(*p) \wedge d\phi + dH + \lambda d\phi j(*1) \\ &= [d(*p) - m^2\phi(*1) - \lambda j(*1)] \wedge [p - d\phi]. \end{aligned}$$

No derivative of j appears because that would involve five factors of dt ’s and there can be only four such factors in a wedge product. The equations for a

solution $t \rightarrow \phi = F(t), p_\nu = G_\nu$ become

$$G_\nu = F_{,\nu}, \quad (\partial^\mu \partial_\mu - m^2)F = \lambda j.$$

The modification caused by λB as $\lambda \rightarrow 0$ to the solution F_0 is given by the retarded solution to the inhomogeneous Klein-Gordon equation,

$$F(t) = F_0(t) + \lambda K(t), \quad G_\nu = F_{,\nu}$$

where

$$K(t) = \int G_R(t-s)j(s)d^4s.$$

The retarded and advanced Green's functions $G_R(t), G_A(t)$ are the unique solutions

$$G_{R,A}(t) = \frac{1}{(2\pi)^4} \int d^4k \frac{\exp(-ik^0 t^0 + i\mathbf{k} \cdot \mathbf{t})}{(k^0 \pm i\epsilon)^2 - \mathbf{k}^2 - m^2}$$

of

$$(\partial^\mu \partial_\mu - m^2)G_R(t) = \delta^4(t)$$

with the boundary condition that $G_R(t)$ is non-zero only in the forward light-cone and $G_A(t)$ in the backward light-cone.

Thus the vertical field is determined to be ($\lambda \rightarrow 0$ can be factored out to give the tangent vector field)

$$Y_B = K(t) \frac{\partial}{\partial \phi} + K_{,\nu} \frac{\partial}{\partial p_\nu}$$

Consider the observable

$$A = \int \alpha = \int \phi k(*1)$$

where $k(t)$ is another switching function. The change in A due to B is given by $D_B A = L_{Y_B}(A)$. Now,

$$\begin{aligned} L_{Y_B}(A) &= \int [i_Y(d\phi k(*1)) + d(\phi k i(Y)(*1))] \\ &= \int k K(*1), \end{aligned}$$

because $i(Y_B)(*1) = 0$. Thus

$$\begin{aligned} D_B A &= \int d^4t k(t) K(t) \\ &= \int \int d^4t d^4s k(t) G_R(t-s) j(s) \end{aligned}$$

Reversing the role of B and A we get the Peierls bracket

$$[A, B] = D_B A - D_A B = \int \int d^4 t d^4 s k(t) \Delta(t-s) j(s)$$

where Δ is the Pauli-Jordan function $\Delta = G_R - G_A$. This is equivalent to the commutator

$$[\phi(t), \phi(s)] = \Delta(t-s)$$

when k and j are Dirac deltas with support at t and s respectively.

The Peierls bracket for the field ϕ and momentum p can be calculated by considering the observable

$$C = \lambda(*p) \wedge l = -\lambda p_\mu l^\mu(*1)$$

where in this case we must employ a 1-form switching function l to smear the momentum. The 5-form is

$$\Omega_C = [d(*p) - m^2 \phi(*1)] \wedge [p + \lambda l - d\phi].$$

The relevant equation for the modified solution is

$$(\partial^\mu \partial_\mu - m^2)F = \lambda \partial^\mu l_\mu$$

because $d(*p)$ becomes $d(*(\phi - l)) = \partial^\mu \partial_\mu \phi - \partial^\mu l_\mu$. The change in B is

$$D_C B = \int \int d^4 t d^4 s j(t) G_R(t-s) (\partial^\mu l_\mu)(s).$$

On the other hand we have already calculated the vertical field for B which gives

$$\begin{aligned} D_B C &= - \int K_{,\mu} l^\mu(*1) \\ &= - \int \int d^4 t d^4 s l^\mu(t) \partial_{t^\mu} G_R(t-s) j(s) \\ &= \int \int d^4 t d^4 s (\partial_\mu l^\mu)(t) G_R(t-s) j(s) \\ &= \int \int d^4 t d^4 s j(t) G_A(t-s) (\partial_\mu l^\mu)(s) \end{aligned}$$

after integrating by parts in the third step. Therefore,

$$[B, C] = \int \int d^4 t d^4 s j(t) \Delta(t-s) (\partial_\mu l^\mu)(s)$$

which, for $j(t) = \delta^4(t)$ and $l_\mu = (1, 0, 0, 0) \delta^4(s)$ gives the equal-time $(t^0 - s^0)$ canonical Poisson bracket of the “3+1” version of field theory

$$[\phi(t, \mathbf{t}), p_0(t, \mathbf{s})] = \delta(\mathbf{t} - \mathbf{s})$$

because

$$\partial_0 \Delta(t) = -\delta^3(\mathbf{t}).$$

6 $i_Y \Theta$ for $Y = v^\mu \partial / \partial t^\mu$

As an illustration of the Noether theorem in our formalism let us evaluate $i_Y \Theta$ for the present scalar field case for spacetime translations. The vector field for constant infinitesimal displacement v^μ is

$$Y = v^\mu \frac{\partial}{\partial t^\mu}$$

We are not assuming that spacetime is flat or that Y are Killing fields of translation symmetry.

We know that

$$\begin{aligned} *p &= p_\mu * (dt^\mu) \\ &= \frac{1}{3!} \sqrt{-g} p_\mu g^{\mu\alpha} \varepsilon_{\alpha\nu\sigma\tau} (\nu\sigma\tau) \\ &\equiv \frac{1}{3!} \sqrt{-g} p^\alpha \varepsilon_{\alpha\nu\sigma\tau} (\nu\sigma\tau) \end{aligned}$$

where we introduce a convenient notation

$$(\nu\sigma\tau) \equiv dt^\nu \wedge dt^\sigma \wedge dt^\tau,$$

with similar notation for two or four factors of dt^μ and we have defined the contravariant canonical momentum

$$p^\mu = g^{\mu\nu} p_\nu.$$

A simple calculation using

$$\begin{aligned} i_Y(dt^\mu \wedge dt^\nu \wedge dt^\sigma) &= v^\mu(dt^\nu \wedge dt^\sigma) - v^\nu(dt^\mu \wedge dt^\sigma) \\ &\quad + v^\sigma(dt^\mu \wedge dt^\nu) \end{aligned}$$

gives,

$$i_Y(*p) = \frac{1}{2!} \sqrt{-g} p^\alpha v^\beta \varepsilon_{\alpha\beta\sigma\tau} (\sigma\tau).$$

We can write this also as

$$i_Y(*p) = p_\mu v_\nu * (dt^\mu \wedge dt^\nu) = *(p \wedge Y^\flat)$$

where $v_\mu = g_{\mu\nu} v^\nu$ and $Y^\flat = v_\nu dt^\nu$ is the covariant field corresponding to Y after lowering the index by the metric.

As Y involves $\partial / \partial t^\mu$ whose action on $d\phi$ is zero

$$i_Y(*p \wedge d\phi) = (i_Y * p) \wedge d\phi,$$

and

$$\begin{aligned} i_Y[*p \wedge p] &= [i_Y * p] \wedge p - *p(i_Y p) \\ &= *(p \wedge Y^\flat) \wedge p - p(Y) * p. \end{aligned}$$

The formula $i_Y(*p) = *(p \wedge Y^\flat)$, although elegant, is not very useful for calculations. A straightforward expression for $i_Y(*p) \wedge p$ is

$$i_Y(*p) \wedge p = [p_\mu(p.v) - v_\mu(p.p)] * (dt^\mu)$$

where

$$p.v = p_\mu v^\mu = \langle p, Y^\flat \rangle, \quad p.p = p_\mu p^\mu = \langle p, p \rangle.$$

Thus the calculation of $i_Y \Theta$ proceeds as follows,

$$\begin{aligned} i_Y \Theta &= i_Y \left[*p \wedge d\phi - \frac{1}{2} *p \wedge p - \frac{1}{2} m^2 \phi^2 * (1) \right] \\ &= (i_Y *p) \wedge \left(d\phi - \frac{1}{2} p \right) + \frac{1}{2} p(Y) *p \\ &\quad - \frac{1}{2} m^2 \phi^2 i_Y * (1) \end{aligned}$$

Evaluating it “on-shell” means we can put $p = d\phi$. Using expression for $i_Y(*p) \wedge p$, $p(Y) = p.v$ and the fact that

$$\begin{aligned} i_Y * (1) &= \sqrt{-g}(v^0[123] - v^1[023] + v^2[013] - v^3[012]) \\ &= v_\mu * (dt^\mu), \end{aligned}$$

we get

$$\begin{aligned} i_Y \Theta &= \left(\frac{1}{2} [p_\mu(p.v) - v_\mu(p.p)] + \frac{1}{2} (p.v) p_\mu \right) * (dt^\mu) \\ &\quad - \frac{1}{2} m^2 \phi^2 v_\mu * (dt^\mu) \\ &= \left(p_\mu(p.v) - \frac{1}{2} [(p.p) + m^2 \phi^2] v_\mu \right) * (dt^\mu) \Big|_0 \\ &= \langle d\phi, Y^\flat \rangle (*d\phi) - \frac{1}{2} [\langle d\phi, d\phi \rangle + m^2 \phi^2] (*Y^\flat) \end{aligned}$$

which can also be written in the useful form

$$i_Y \Theta = \left[\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} (g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + m^2 \phi^2) \right] v^\mu * (dt^\nu) \quad (19)$$

7 Examples of conserved quantities

As an illustration we calculate the conserved quantities for the Klein-Gordon field in Minkowski background. In this case $L_Y \Theta = 0$ (actually $L_Y(*p \wedge d\phi) = 0$ and $L_Y H = 0$ independently) for any of the ten Killing vector fields Y corresponding to Poincare transformations. For spacetime translations we have

derived a formula in the last section. Since we usually integrate on the spacelike surface $t^0 = \text{constant}$, it is enough to calculate the term $*(dt^0) = -(123)$, which alone will give a non-zero contribution on $t = \text{constant}$ surface. The following table gives the expected conserved quantities (energy and momentum densities) for time- and space-translations

Y	v^μ	$-(123)$ part of $i_Y \Theta$
$\partial/\partial t^0$	$(1, 0, 0, 0)$	$(1/2)[(\phi_{,0})^2 + (\nabla\phi)^2 + m^2\phi^2]d^3t$
$\partial/\partial t^1$	$(0, 1, 0, 0)$	$[\phi_{,1}\phi_{,0}]d^3t$

A Notation

The spacetime is a Riemannian space with coordinates $t^\mu, \mu = 0, 1, 2, 3$. Basis vectors in a tangent space are written $\partial_\mu = \partial/\partial t^\mu$. The metric is given by the inner product $\langle \partial_\mu, \partial_\nu \rangle = g_{\mu\nu}$. The cotangent spaces have basis elements dt^μ with $\langle dt^\mu, dt^\nu \rangle = g^{\mu\nu}$. The metric has signature $(-1, 1, 1, 1)$. The wedge product is defined so that $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ for one-forms α and β . The exterior derivative is defined so that for an r -form $\alpha = a_{\mu_1 \dots \mu_r} dt^{\mu_1} \wedge \dots \wedge dt^{\mu_r}$ the derivative is the $(r+1)$ -form

$$d\alpha = a_{\mu_1 \dots \mu_r, \nu} dt^\nu \wedge dt^{\mu_1} \wedge \dots \wedge dt^{\mu_r}.$$

The Hodge star is a linear operator that maps r -forms into $(4-r)$ -forms in our four-dimensional space. The definition is

$$\begin{aligned} *(dt^{\mu_1} \wedge \dots \wedge dt^{\mu_r}) &= [(4-r)!]^{-1} \sqrt{-g} g^{\mu_1 \nu_1} \dots \\ &\quad g^{\mu_r \nu_r} \varepsilon_{\nu_1 \dots \nu_r \nu_{r+1} \dots \nu_4} dt^{\nu_{r+1}} \dots dt^{\nu_4} \end{aligned}$$

where g denotes the determinant of $g_{\mu\nu}$ and ε is the antisymmetric tensor defined with $\varepsilon_{0123} = 1$. The one-dimensional space of 0-forms has the unit vector equal to real number 1. The one-dimensional space space of 4-forms has the chosen orientation given by the unit vector $\varepsilon = n^0 \wedge n^1 \wedge n^2 \wedge n^3$ where n^μ are the orthonormal basis vectors. In ordinary basis $\varepsilon = \sqrt{g} dt^0 \wedge dt^1 \wedge dt^2 \wedge dt^3$. The star operator acting on the zero form equal to constant number 1 is denoted by $*1 = \varepsilon = \sqrt{g} dt^0 \wedge dt^1 \wedge dt^2 \wedge dt^3$. We have the simple result that $dt^\mu \wedge *dt^\nu = -*dt^\nu \wedge dt^\mu = g^{\mu\nu}(*1)$

Note carefully that $*1$ is not the same as $*(1)$ where the shorthand notation (μ) is used for dt^μ . Similarly we use (12) for $dt^1 \wedge dt^2$, (013) for $dt^0 \wedge dt^1 \wedge dt^3$ etc.

The interior product $i(X)$ of a vector X with an r -form α gives an $(r-1)$ -form $i(X)\alpha$ defined by

$$(i(X)\alpha)(Y_1, \dots, Y_{r-1}) = \alpha(X, Y_1, \dots, Y_{r-1})$$

When it is more convenient we will denote the interior product operator by i_X in place of $i(X)$.

Two successive applications of interior products on a form will be denoted by

$$i(X, Y)\alpha \equiv [i(X) \circ i(Y)]\alpha = i(X)[i(Y)\alpha]$$

Note that $i(X, Y) = -i(Y, X)$. Similarly successive applications $i(XY \dots Z)$ of many such interior products can be defined. If α is an r -form then

$$i(X)(\alpha \wedge \beta) = [i(X)\alpha] \wedge \beta + (-1)^r \alpha \wedge i(X)\beta$$

In order to abbreviate expressions we use $i(12)$ for $i(X_1 X_2) = i(X_1) \circ i(X_2)$ etc. when there is no confusion.

B Solution submanifold of Ω

We give the calculation of $i(X_3 X_2 X_1 X_0)\Omega$ for $H = (*p) \wedge p/2 + m^2 \phi^2 * (1)/2$ in Minkowski space for illustration. We take the independent tangent vectors to the section

$$\sigma : t \rightarrow (t^\mu, \phi = F(t), p_\nu = G_\nu(t))$$

the push-forwards

$$X_\mu \equiv \sigma_*(\partial_\mu) = \partial_\mu + F_{,\mu} \partial_\phi + G_{\nu,\mu} \partial_{p_\nu}$$

The calculation involves the following expressions (we use abbreviations of Appendix A) :

$$\begin{aligned} *1 &= (0123) \\ dt^\mu &= [-(123), -(023), +(013), -(012)] \end{aligned}$$

$$\begin{aligned} d(p_\mu * dt^\mu) &= dp_\mu * dt^\mu = -dp_0(123) - dp_1(023) + dp_2(013) - dp_3(012) \\ i(0)(0123) &= (123), \\ i(1)(0123) &= -(023), \\ i(2)(0123) &= (013), \\ i(3)(0123) &= -(012) \end{aligned}$$

$$\begin{aligned} i(0)(d * p) &= dp_1(23) - dp_2(13) + dp_3(12) - G_{0,0}(123) - G_{1,0}(023) \\ &\quad + G_{2,0}(013) - G_{3,0}(012) \\ i(1)(d * p) &= dp_0(23) + dp_2(03) - dp_3(02) - G_{0,1}(123) - G_{1,1}(023) \\ &\quad + G_{2,1}(013) - G_{3,1}(012) \\ i(2)(d * p) &= -dp_0(23) - dp_1(03) + dp_3(01) - G_{0,2}(123) - G_{1,2}(023) \\ &\quad + G_{2,2}(013) - G_{3,2}(012) \end{aligned}$$

$$\begin{aligned}
i(0)(d * p) &= dp_0(12) + dp_1(02) - dp_2(01) - G_{0,3}(123) - G_{1,3}(023) \\
&\quad + G_{2,3}(013) - G_{3,3}(012) \\
i(10)(d * p) &= dp_2(3) - dp_3(2) - G_{0,0}(23) - G_{2,0}(03) + G_{3,0}(02) \\
&\quad + G_{1,1}(23) - G_{2,1}(13) + G_{3,1}(12) \\
i(20)(d * p) &= -dp_1(3) + dp_3(1) + G_{0,0}(13) + G_{1,0}(03) - G_{3,0}(01) \\
&\quad + G_{1,2}(23) - G_{2,2}(13) + G_{3,2}(12) \\
i(30)(d * p) &= dp_1(2) - dp_2(1) - G_{0,0}(12) - G_{1,0}(02) + G_{2,0}(01) \\
&\quad + G_{1,3}(23) - G_{2,3}(13) + G_{3,3}(12) \\
i(21)(d * p) &= -dp_0(3) - dp_3(0) + G_{0,1}(13) + G_{1,1}(03) - G_{3,1}(01) \\
&\quad + G_{0,2}(23) + G_{2,2}(03) - G_{3,2}(02) \\
i(31)(d * p) &= dp_0(2) + dp_2(0) - G_{0,1}(12) - G_{1,1}(02) + G_{2,1}(01) \\
&\quad + G_{0,3}(23) + G_{2,3}(03) - G_{3,3}(02) \\
i(32)(d * p) &= -dp_0(1) - dp_1(0) - G_{0,2}(12) - G_{1,2}(02) + G_{2,2}(01) \\
&\quad - G_{0,3}(13) - G_{1,3}(03) + G_{3,3}(01) \\
i(321)(d * p) &= dp_0 - G_{0,1}(1) - G_{1,1}(0) - G_{0,2}(2) - G_{2,2}(0) \\
&\quad - G_{0,3}(3) - G_{3,3}(0) \\
i(320)(d * p) &= dp_1 - G_{0,0}(1) - G_{1,0}(0) - G_{1,2}(2) + G_{2,2}(1) \\
&\quad - G_{1,3}(3) + G_{3,3}(1) \\
i(310)(d * p) &= -dp_2 + G_{0,0}(2) + G_{2,1}(0) - G_{1,1}(2) + G_{2,1}(1) \\
&\quad + G_{2,3}(3) - G_{3,3}(2) \\
i(210)(d * p) &= dp_3 - G_{0,0}(3) - G_{3,0}(0) + G_{1,1}(3) - G_{3,1}(1) \\
&\quad + G_{2,2}(3) - G_{3,2}(2) \\
i(3210)(d * p) &= -G_{0,0} + G_{1,1} + G_{2,2} + G_{3,3}
\end{aligned}$$

If A is a 4-form and B a 1-form then

$$\begin{aligned}
i(3210)[A \wedge B] &= [i(3210)A]B + [i(321)A]i(0)B - [i(320)A]i(1)B \\
&\quad + [i(310)A]i(2)B - [i(210)A]i(3)B
\end{aligned}$$

For $A = d * p - m^2 \phi * 1$ and $B = p - d\phi$ the expression for $i(3210)[A \wedge B] = i(3210)\Omega$ is a 1-form in the extended phase space which should be equated to zero. The coefficients of dp_μ equated to zero give $G_\mu - F_{,\mu} = 0$ and the coefficient of $d\phi$ gives $-G_{0,0} + G_{1,1} + G_{2,2} + G_{3,3} = 0$. These imply the Klein-Gordon equation for F .

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